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## Radial Points in the Plane

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A *radial point* for a finite set  $P$  in the plane is a point  $q \notin P$  with the property that each line connecting  $q$  to a point of  $P$  passes through at least one other element of  $P$ . We prove a conjecture of Pinchasi, by showing that the number of radial points for a non-collinear  $n$ -element set  $P$  is  $O(n)$ . We also present several extensions of this result, generalizing theorems of Beck, Szemerédi and Trotter, and Elekes on the structure of incidences between points and lines.

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### 1. INTRODUCTION

Let  $P$  be a set of  $n$  points in the plane, not all lying on the same line. A point  $q \notin P$  is called a *radial point* (for  $P$ ) if for every line  $\ell$  passing through  $q$  we have  $|\ell \cap P| \neq 1$ . In other words, every line connecting  $q$  to some point  $p \in P$  passes through at least one other element of  $P$ .

For instance, let  $P$  be the vertex set of a regular  $2k$ -gon in the plane. Then, the intersection of the line at infinity with each line supporting an edge of  $P$  is a radial point for  $P$ . The center of the regular  $2k$ -gon is another radial point. We thus have a  $2k$ -element set which has  $k + 1$  radial points.

Pinchasi [6] conjectured that any non-collinear set of  $n$  points in the plane has at most  $O(n)$  radial points. He verified this conjecture in the special case when no three points of the set are collinear, and he also established the weaker upper bound  $O(n^{3/2})$  for the general case. The main result of our paper is a proof of Pinchasi's conjecture.

**THEOREM 1.** *The maximum possible number of radial points for a non-collinear set of  $n$  points in the plane is  $\Theta(n)$ .*

The construction of Pinchasi depicted in Figure 1 shows that the constant of proportionality in Theorem 1 is at least  $5/3$  (although it is still possible that the constant is smaller if  $n > 6$  is sufficiently large). Nevertheless, Pinchasi has shown that the number of radial points of a non-collinear set  $P$  of  $n$  points in the plane that lies in a halfplane disjoint from  $P$  is at most  $0.9n$ . The constant yielded by our proof is much larger, though.

The notion of radially can be extended, as follows. For any  $\varepsilon \in (0, 1)$ , we say that  $q \notin P$  is an  $\varepsilon$ -radial point for  $P$ , if the number of distinct lines connecting  $q$  to the points of  $P$  is at most  $(1 - \varepsilon)n$ .

Note that, according to this definition, any radial point is  $(1/2)$ -radial. If at least  $\varepsilon n + 1$  points of  $P$  lie on a common line, then every point of this line (which does not belong to  $P$ ) is  $\varepsilon$ -radial. Therefore, in this case the number of  $\varepsilon$ -radial points for  $P$  is infinite. However, by a slight modification of the proof of Theorem 1, we can obtain the following positive result.

**THEOREM 2.** *For any  $0 < \delta < \varepsilon < 1$ , there exists a constant  $C = C(\delta, \varepsilon)$  with the following property.*

*Let  $P$  be any set of  $n$  points in the plane. Then the number of  $\varepsilon$ -radial points that do not belong to any line passing through at least  $\delta n$  elements of  $P$  is at most  $Cn$ .*

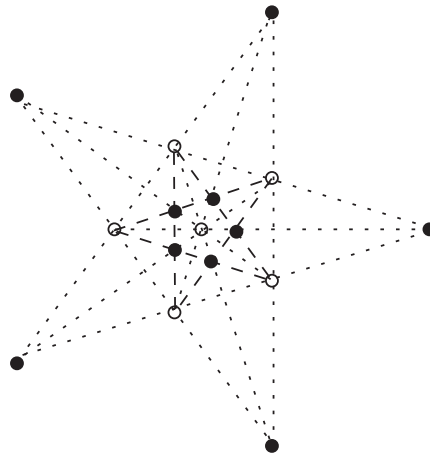


FIGURE 1. Pinchasi's set of six points with ten radial points.

COROLLARY 3. *For any  $0 < \delta < \varepsilon < 1$ , there exists a constant  $C = C(\delta, \varepsilon)$  with the following property.*

*Let  $P$  be any set of  $n$  points in the plane, no  $\delta n$  of which are collinear. Then the number of  $\varepsilon$ -radial points for  $P$  is at most  $Cn$ .*

The following result of Elekes [3] (see also [4]) is another immediate consequence of Theorem 2.

COROLLARY 4. *For every  $\varepsilon \leq 1$  there is a constant  $C = C(\varepsilon)$  such that every  $n$ -element set  $P$  on a line has at most  $Cn$  subsets similar to a given  $\lfloor \varepsilon n \rfloor$ -element set  $Q$ .*

PROOF. Assume without loss of generality that  $P$  and  $Q$  lie on two parallel lines in the plane. For any similar copy  $Q' \subset P$  of  $Q$ , the lines connecting the corresponding points of  $Q$  and  $Q'$  are concurrent. Their common point is some  $\varepsilon/2$ -radial point for  $P \cup Q$ , which does not belong to any line containing more than two elements of this set. Hence, Theorem 2 can be applied to bound the number of such points (and sets  $Q'$ ).  $\square$

The proof of Theorem 1 is presented in Section 2. In Section 3, we establish a generalization of Theorem 2. The main result of that section, Theorem 3.1, is a strengthening of the following theorem of Beck [1] and Szemerédi and Trotter [9] (also known as the 'weak Dirac–Motzkin conjecture').

COROLLARY 5. *There is a constant  $c > 0$  with the property that any non-collinear set  $P$  of points in the plane has an element  $q$  such that the number of distinct lines connecting  $q$  to all other points of  $P$  is at least  $c|P|$ .*

According to the 'strong Dirac–Motzkin conjecture', this statement should be true with  $c = 1/2$ . Corollary 5 has several interesting applications in combinatorial geometry (see, e.g., [7] for the most recent one).

## 2. PROOF OF THEOREM 1

Clearly, we only need to prove the upper bound. The idea of the proof is the following. Let  $q$  be a radial point for  $P$ , and let  $j$  denote the average number of points of  $P$  that lie on the lines

connecting  $q$  to the points of  $P$ . If each of these lines contained exactly  $j$  points, the number of lines would be about  $n/j$ . In general, some relaxation of this relationship is needed: we show that any radial point  $q$  has an ‘index’  $j$  such that the number of lines that connect  $q$  to at least  $j$  points of  $P$  is at least  $n/(6j \log^2 j)$ . We then prove (in Lemma 2.4) that the number of radial points that have a small index (up to  $c\sqrt{n}$  for some constant  $c$ ) is linear, and finally derive a linear bound on the number of radial points with a large index.

The following well known results of Szemerédi and Trotter [9, 10] are crucial to the proof.

LEMMA 2.1. (i) *The number of incidences between  $l$  distinct lines and  $n$  distinct points in the plane cannot exceed  $3n^{2/3}l^{2/3} + n + l$ .*

(ii) *For any  $j \leq n$ , the number of lines containing at least  $j$  elements of a given set of  $n$  points in the plane cannot exceed  $40 \cdot \max\{n^2/j^3, n/j\}$ .*

Part (i) is asymptotically tight in the worst case, apart from the values of the constants, and it implies part (ii). The best known constants for (i) are given in [5]. A simple proof of (i) was found by Székely [8]; see also [2].

In the following, let  $P$  be a fixed set of  $n$  points in the plane, not all on a line, and let  $R$  be the set of radial points for  $P$ .

Let  $\mathcal{L}$  be the set of all lines that pass through at least two points of  $P$ . Denote by  $\mathcal{L}_j$  (resp.  $\mathcal{L}_{\leq j}$ ,  $\mathcal{L}_{\geq j}$ ) the set of those elements of  $\mathcal{L}$  which contain precisely  $j$  (resp. at most  $j$ , at least  $j$ ) elements of  $P$ . Throughout this paper, we write  $\log j$  for  $\ln j$ , the natural logarithm of  $j$ .

LEMMA 2.2. *For every radial point  $q \in R$ , there is an integer  $2 \leq j \leq n-2$  such that the number of lines in  $\mathcal{L}_{\geq j}$  passing through  $q$  is at least  $\lceil \frac{n}{3j \log^2 j} \rceil$ .*

PROOF. Suppose, to the contrary, that for some  $q \in R$  no such  $j$  exists. Let  $l_j$  denote the number of lines in  $\mathcal{L}_j$  passing through  $q$ . We have

$$\begin{aligned} l_2 + l_3 + \cdots + l_{n-2} &< \frac{n}{3 \cdot 2 \log^2 2}, \\ l_3 + \cdots + l_{n-2} &< \frac{n}{3 \cdot 3 \log^2 3}, \\ &\dots \\ l_{n-2} &< \frac{n}{3(n-2) \log^2(n-2)}. \end{aligned}$$

Summing up all these inequalities, the first one with coefficient two, and noting that  $2l_2 + 3l_3 + \cdots + (n-2)l_{n-2} = n$ , we obtain

$$n < n \cdot \left( \frac{1}{3 \log^2 2} + \sum_{j=3}^{n-2} \frac{1}{3j \log^2 j} \right) < 0.8n,$$

the desired contradiction.  $\square$

DEFINITION 2.3. The *index* of a point  $q \in R$  is the smallest integer  $j = j(q)$  such that  $2 \leq j \leq n-2$  and the number of lines in  $\mathcal{L}_{\geq j}$  passing through  $q$  is at least  $\lceil \frac{n}{6j \log^2 j} \rceil$ .

Clearly, Lemma 2.2 implies that every radial point has an index. (The constant 6 has been chosen for technical reasons that will become clear later.)

LEMMA 2.4. *For every  $c > 0$ , there exists  $c' > 0$  such that the number of radial points whose index is at most  $c\sqrt{n}$  does not exceed  $c'n$ .*

PROOF. Note first that Lemma 2.1(ii) implies that for  $j \leq \sqrt{n}$ , the size of  $\mathcal{L}_{\geq j}$  is at most  $40n^2/j^3$ , whereas for  $j > \sqrt{n}$ , the size of  $\mathcal{L}_{\geq j}$  is at most  $40n/j$ . We will assume that  $c > 1$ ; in the other case the proof becomes even simpler.

For  $j = 2, \dots, \lfloor c\sqrt{n} \rfloor$ , let  $R_j$  denote the set of radial points of index  $j$ , and put  $r_j = |R_j|$ . Let  $I_j$  denote the number of incidences between the points in  $R_j$  and the lines in  $\mathcal{L}_{\geq j}$ . Since each  $q \in R_j$  is incident on at least  $\frac{n}{6j \log^2 j}$  lines of  $\mathcal{L}_{\geq j}$ , we have

$$I_j \geq \frac{r_j n}{6j \log^2 j}.$$

Applying Lemma 2.1(i) to  $R_j$  and  $\mathcal{L}_{\geq j}$ , we find that

$$I_j \leq 3r_j^{2/3} |\mathcal{L}_{\geq j}|^{2/3} + r_j + |\mathcal{L}_{\geq j}|.$$

Comparing the last two inequalities, we have

$$\frac{r_j n}{6j \log^2 j} \leq 3r_j^{2/3} \cdot 40^{2/3} \left( \max \left\{ \frac{n^2}{j^3}, \frac{n}{j} \right\} \right)^{2/3} + r_j + 40 \cdot \max \left\{ \frac{n^2}{j^3}, \frac{n}{j} \right\}.$$

Therefore,

$$\frac{r_j n}{6j \log^2 j} \leq \max \left\{ 3^2 r_j^{2/3} \left( \frac{40n^2}{j^3} \right)^{2/3}, 3^2 r_j^{2/3} \left( \frac{40n}{j} \right)^{2/3}, 3r_j, \frac{120n^2}{j^3}, \frac{120n}{j} \right\},$$

which yields, for  $j \leq \sqrt{n}$ ,

$$r_j \leq \max \left\{ 6^3 \cdot 3^6 \cdot 40^2 \frac{n \log^6 j}{j^3}, 720 \frac{n \log^2 j}{j^2} \right\} \leq 10^9 \frac{n \log^2 j}{j^2},$$

and, for  $j > \sqrt{n}$ ,

$$r_j \leq \max \left\{ 6^3 \cdot 3^6 \cdot 40^2 \frac{j \log^6 j}{n}, 720 \log^2 j \right\} \leq 720 \log^2 j,$$

provided that  $n$  is at least some sufficiently large constant  $n_0$ . Summing up these inequalities, we obtain that the number of radial points with index at most  $c\sqrt{n}$  satisfies (for  $n > n_0$ )

$$\sum_{j=2}^{\lfloor c\sqrt{n} \rfloor} r_j \leq \sum_{j=2}^{\lfloor \sqrt{n} \rfloor} 10^9 \frac{n \log^2 j}{j^2} + \sum_{j=\lfloor \sqrt{n} \rfloor+1}^{\lfloor c\sqrt{n} \rfloor} 720 \log^2 j \leq c'n,$$

for an appropriate constant  $c'$ . (For  $n \leq n_0$ , this will trivially hold, if we choose  $c'$  sufficiently large.)  $\square$

Note that the dependence of  $c'$  on  $c$  is rather weak. In fact, if  $n$  is at least some constant  $n_0(c)$  that depends on  $c$ , we can choose  $c'$  to be an absolute constant independent of  $c$ .

Let  $R^*$  denote the set of all radial points with index greater than  $c\sqrt{n}$ , where  $c > 1$  is a constant to be specified later, and let  $\mathcal{L}^* = \mathcal{L}_{\geq c\sqrt{n}}$ . Then  $R^* = R_1^* \cup R_2^*$ , where  $R_1^*$  (resp.  $R_2^*$ ) denotes the set of those elements of  $R^*$  that lie on exactly one line (resp. at least two lines) belonging to  $\mathcal{L}^*$ .

The number of incidences between the original point set  $P$  and  $\mathcal{L}^*$  is at least  $c\sqrt{n}|\mathcal{L}^*|$ . On the other hand, by Lemma 2.1(i), the same quantity can be bounded from above by  $3n^{2/3}|\mathcal{L}^*|^{2/3} + n + |\mathcal{L}^*|$ . Thus,  $|\mathcal{L}^*| < \sqrt{n}$ , provided that  $c$  is sufficiently large ( $c \geq 5$  suffices). This immediately implies that

$$|R_2^*| \leq \binom{|\mathcal{L}^*|}{2} < n/2.$$

It remains to show that the size of  $R_1^*$  is  $O(n)$ .

LEMMA 2.5. *For every point  $q \in R_1^*$ , the (unique) line in  $\mathcal{L}^*$  passing through  $q$  contains more than  $\lceil n/2 \rceil$  elements of  $P$ .*

PROOF. Assume, in order to obtain a contradiction, that some  $q \in R_1^*$  violates this condition. Then each line through  $q$ , different from the unique element  $\ell^*$  of  $\mathcal{L}^*$  that passes through  $q$ , contains at most  $c\sqrt{n}$  points of  $P$ . (At this point we use the fact that the constant in the denominator of the quantity that appears in Definition 2.3 is 6.) Apply Lemma 2.2 to  $q$  and to the set  $P' = P \setminus \ell^*$ , to conclude that there exists an integer  $2 \leq j \leq |P'| - 2$  such that  $q$  is incident on at least  $|P'|/(3j \log^2 j)$  lines, each containing at least  $j$  points of  $P'$ , and thus at least  $j$  points of  $P$ . Clearly, we have  $j \leq c\sqrt{n}$ . Since  $|P'| > n/2$ , it follows, according to Definition 2.3, that the index of  $q$  (with respect to  $P$ ) is at most  $j \leq c\sqrt{n}$ . That is,  $q \notin R^*$ , a contradiction.  $\square$

Now we are in a position to complete the proof of Theorem 1. Since there are no two distinct lines passing through more than  $\lceil n/2 \rceil$  points of  $P$ , it follows from Lemma 2.5 that all points of  $R_1^*$  are collinear. Let  $\ell$  be the line containing  $R_1^*$ .

Let  $P'$  denote the set of points of  $P$  that do not lie on  $\ell$ . For each radial point  $q \in R_1^*$ , let  $f(q)$  denote the number of pairs  $\{p', p''\} \subset P'$ , for which  $q, p'$ , and  $p''$  are collinear. Obviously,

$$\sum_{q \in R_1^*} f(q) \leq \binom{|P'|}{2}.$$

On the other hand,  $f(q) \geq |P'|/2$  holds for every  $q \in R_1^*$ . This immediately implies that

$$|R_1^*| \leq |P'| - 1 < n/2,$$

completing the proof of Theorem 1.

### 3. $\varepsilon$ -RADIAL AND QUASIRADIAL POINTS

If a point  $q \notin P$  is  $\varepsilon$ -radial, then the total number of points on those lines which pass through  $q$  and contain more than one element of  $P$  is at least  $\varepsilon n$ . This suggests the following definition.

Let  $P$  be a set of  $n$  points in the plane, and let  $q$  be another point which may or may not belong to  $P$ . For any positive real  $\varepsilon \leq 1$  and for any integer  $k \geq 2$ , we say that  $q$  is an  $(\varepsilon, k)$ -quasiradial point for  $P$ , if the number of points  $p \in P \setminus \{q\}$  sitting on lines that pass through  $q$  and at least  $k$  elements of  $P \setminus \{q\}$  is at least  $\varepsilon n$ .

THEOREM 3.1. *For any  $0 < \delta < \varepsilon \leq 1$  and for any integer  $k \geq 2$ , there exists a constant  $C = C(\delta, \varepsilon, k)$  with the following property.*

Let  $P$  be any set of  $n$  points in the plane. Remove from the plane all lines that pass through at least  $\delta n$  elements of  $P$ . Then the total number of  $(\varepsilon, k)$ -quasiradial points lying in the remaining regions is at most  $Cn$ , provided that  $n$  is large enough. Moreover, as  $n$  tends to infinity,

$$C(\delta, \varepsilon, k) = O\left(\frac{1}{(\varepsilon - \delta)^3 k \log k}\right).$$

The proof of this theorem is similar to that of Theorem 1.

Let  $P$  be a fixed set of  $n$  points, and let  $Q$  denote the set of all  $(\varepsilon, k)$ -quasiradial points for  $P$ . Then  $Q = Q_0 \cup Q_1$ , where  $Q_0 = Q \setminus P$  and  $Q_1 = Q \cap P$ . Instead of Lemma 2.2, we now have the following lemma.

LEMMA 3.2. *For every point  $q \in Q_0$ , there is an integer  $j \geq k$ , and, for every point  $q \in Q_1$ , there is an integer  $j \geq k + 1$  such that the number of lines in  $\mathcal{L}_{\geq j}$  passing through  $q$  is at least  $\lceil \frac{\varepsilon \log k}{3j \log^2 j} n \rceil$ .*

PROOF. Suppose, to the contrary, that for some  $q \in Q$  no such  $j$  exists. Let  $l_j$  denote the number of lines in  $\mathcal{L}_j$  passing through  $q$ , for  $1 \leq j \leq n$ . We have

$$\begin{aligned} l_{k+1} + l_{k+2} + \cdots + l_n &< \frac{\varepsilon \log k}{3(k+1) \log^2(k+1)} \cdot n, \\ l_{k+2} + \cdots + l_n &< \frac{\varepsilon \log k}{3(k+2) \log^2(k+2)} \cdot n, \\ &\dots \\ l_n &< \frac{\varepsilon \log k}{3 \cdot n \log^2 n} \cdot n, \end{aligned}$$

and, if  $q \in Q_0$ , then also

$$l_k + l_{k+1} + \cdots + l_n < \frac{\varepsilon \log k}{3k \log^2 k} \cdot n.$$

If  $q \in Q_0$ , then summing up these inequalities (the last one with coefficient  $k$ ), and comparing it with the relation  $kl_k + (k+1)l_{k+1} + \cdots \geq \varepsilon n$ , we obtain

$$\varepsilon n < \varepsilon \log k \cdot \left( \frac{1}{3 \log^2 k} + \sum_{j=k+1}^n \frac{1}{3j \log^2 j} \right) \cdot n < 0.8\varepsilon n,$$

the desired contradiction. If  $q \in Q_1$ , then  $kl_{k+1} + (k+1)l_{k+2} + \cdots \geq \varepsilon n$ . Hence, arguing as before, we now have

$$\varepsilon n < \varepsilon \log k \cdot \left( \frac{k}{3(k+1) \log^2(k+1)} + \sum_{j=k+2}^n \frac{1}{3j \log^2 j} \right) \cdot n < 0.8\varepsilon n,$$

again a contradiction. □

DEFINITION 3.3. The *index* of a point  $q \in Q_0$  (resp.  $q \in Q_1$ ) is the smallest integer  $j = j(q) \geq k$  (resp.  $j = j(q) \geq k + 1$ ) such that the number of lines in  $\mathcal{L}_{\geq j}$  passing through  $q$  is at least  $\lceil \frac{(\varepsilon - \delta) \log k}{3j \log^2 j} \cdot n \rceil$ .

Lemma 3.2 implies that every point  $q \in Q$  has a (unique) index.

LEMMA 3.4. *Let  $c > 1$  be fixed. There exists an absolute constant  $d > 0$  (independent of  $c, \delta, \varepsilon$ , and  $k$ ) such that the number of points  $q \in Q$  whose index is at most  $c\sqrt{n}$  does not exceed  $\frac{dn}{(\varepsilon-\delta)^3 k \log k}$ , provided that  $n \geq n_0(c, \delta, \varepsilon, k)$  is sufficiently large.*

PROOF. For  $j \geq k$ , let  $R_j \subseteq Q$  now denote the set of  $(\varepsilon, k)$ -quasiradial points of index  $j$ , and put  $r_j = |R_j|$ . Let  $I_j$  denote the number of incidences between the points in  $R_j$  and the lines in  $\mathcal{L}_{\geq j}$ . Since each  $q \in R_j$  is incident on at least  $\frac{(\varepsilon-\delta) \log k}{3j \log^2 j} \cdot n$  lines of  $\mathcal{L}_{\geq j}$ , we have

$$I_j \geq \frac{r_j(\varepsilon - \delta) \log k}{3j \log^2 j} \cdot n,$$

and, by Lemma 2.1(i),

$$I_j \leq 3r_j^{2/3} |\mathcal{L}_{\geq j}|^{2/3} + r_j + |\mathcal{L}_{\geq j}|.$$

Comparing the last two inequalities, as in the proof of Lemma 2.4, an easy computation shows that

$$\sum_{j=k}^{\lfloor c\sqrt{n} \rfloor} r_j \leq \sum_{j=k}^{\lfloor \sqrt{n} \rfloor} \frac{10^9 \log^2 j}{j^2 (\varepsilon - \delta)^3 \log^3 k} \cdot n + \sum_{j=\lfloor \sqrt{n} \rfloor + 1}^{\lfloor c\sqrt{n} \rfloor} \frac{720 \log^2 j}{(\varepsilon - \delta) \log k} \leq \frac{dn}{(\varepsilon - \delta)^3 k \log k},$$

for an appropriate absolute constant  $d$ , provided that  $n$  is sufficiently large.  $\square$

It is interesting to note that we have not excluded in the proof of the lemma points that lie on ‘heavy’ lines, as prescribed in the statement of Theorem 3.1. In particular, the number of quasiradial points with a ‘small’ index is finite.

Let  $Q^*$  denote the set of all elements of  $Q$  which do not belong to any line containing at least  $\delta n$  points of  $P$  and whose index is greater than  $c\sqrt{n}$ , where we set

$$c = \max\{20, (\varepsilon - \delta)^{3/2} \sqrt{(5k/d) \log k}\},$$

with  $d$  being the same constant as in the previous lemma. Let  $\mathcal{L}^* = \mathcal{L}_{\geq c\sqrt{n}}$ . Then  $Q^* = Q_1^* \cup Q_2^*$ , where  $Q_1^*$  (resp.  $Q_2^*$ ) denotes the set of those elements of  $Q^*$  which lie on exactly one line (resp. at least two lines) belonging to  $\mathcal{L}^*$ .

The number of incidences between the original point set  $P$  and  $\mathcal{L}^*$  is at least  $c\sqrt{n}|\mathcal{L}^*|$ . On the other hand, by Lemma 2.1(i), the same quantity can be bounded from above by  $3n^{2/3}|\mathcal{L}^*|^{2/3} + n + |\mathcal{L}^*|$ . Thus,  $|\mathcal{L}^*| < \sqrt{n}/c$ . This immediately implies that

$$|Q_2^*| \leq \binom{|\mathcal{L}^*|}{2} < \frac{dn}{(\varepsilon - \delta)^3 k \log k},$$

if  $n$  is sufficiently large.

To complete the proof of Theorem 3.1, it is sufficient to establish the following.

LEMMA 3.5.  $Q_1^*$  is empty.

PROOF. Suppose, to the contrary, that  $Q_1^*$  has a point  $q$ . Let  $\ell^*$  denote the unique line in  $\mathcal{L}^*$  that passes through  $q$ , and let  $P'$  denote the set of points of  $P$  that lie outside  $\ell^*$ . By assumption,  $\ell^*$  contains at most  $\delta n$  points of  $P$ , so the size of  $P'$  is at least  $(1 - \delta)n$ . Clearly,  $q$  is an  $(\frac{(\varepsilon-\delta)n}{|P'|}, k)$ -quasiradial point for  $P'$ . Applying Lemma 3.2 to  $P'$  and to  $q$ , we find that there is a  $j \geq k$  such that the number of lines in  $\mathcal{L}_{\geq j}$  passing through  $q$  (and excluding  $\ell^*$ ) is at least

$$\left\lceil \frac{(\varepsilon - \delta) \log k}{3j \log^2 j} \cdot n \right\rceil.$$

Hence, by Definition 3.3, the index of  $q$  is at most  $j$ . Since  $q \in Q_1^*$ , every line through  $q$ , except for  $\ell^*$ , contains at most  $c\sqrt{n}$  points of  $P$ . Thus,  $j$  and, therefore, the index of  $q$  cannot exceed  $c\sqrt{n}$ , contradicting our assumption that the index of all points in  $Q^*$  is larger than  $c\sqrt{n}$ .  $\square$

## 4. CONCLUDING REMARKS

**4.1. Super-radial points.** Let  $P$  be a set of  $n$  points in the plane, not all on a line. We call a point  $q \notin P$   $k$ -super-radial for  $P$  if every line that passes through  $q$  and at least one point of  $P$  contains at least  $k$  elements of  $P$ . In this terminology, a radial point for  $P$  can also be called 2-super-radial. Using the definition in the last section, a point is  $k$ -super-radial if and only if it is  $(1, k)$ -quasiradial.

It follows from Theorem 3.1 that the number of  $k$ -super-radial points for  $P$  is at most  $c_k n$ , with a coefficient  $c_k$  that goes to zero as  $k$  increases. The following construction shows that the number of  $k$ -super-radial points can exceed  $c'_k \sqrt{n}$  for some  $c'_k > 0$ .

For any positive integer  $m$ , let  $v_1^m, v_2^m, \dots, v_M^m$  be the sequence of all vectors  $(p, q)$  with relatively prime integer coordinates satisfying  $|p|, |q| \leq m$ , listed in increasing order of their slopes (i.e., according to the clockwise angle between the positive  $x$ -axis and  $(p, q)$ ). It is well known that  $M > cm^2$  for a suitable constant  $c > 0$ . The points  $w_j^m = \sum_{i=1}^j v_i^m$  ( $j = 1, 2, \dots, M$ ) form the vertex set of a centrally symmetric convex polygon  $Q$  of perimeter at most  $M\sqrt{2}m < 4\sqrt{2}m^3$ . Let  $P$  be the set of all points  $(x, y) \in Q$ , whose distance from the boundary of  $Q$  is at most  $2m$ , and  $(k-1)x$  and  $(k-1)y$  are integers. Clearly,  $|P| < 2(k-1)^2 M(2m) < c'(k-1)^2 m^4$ , for some  $c' > 0$ . On the other hand, for every  $i$  ( $1 \leq i \leq M$ ), the intersection of the supporting line of  $v_i^m$  with the line at infinity is easily seen to be a  $k$ -super-radial point for  $P$ . Therefore, the number of  $k$ -super-radial points is at least

$$M > cm^2 > \frac{c''}{k-1} \sqrt{|P|},$$

for an appropriate constant  $c'' > 0$ , as asserted.

**4.2. Lower bounds for quasiradial points.** It can be shown by a similar construction that, for every fixed  $k$  and for every sufficiently large  $n$ , there exists a set  $P$  of  $n$  points in the plane with at most  $o(n)$  points lying on any line, such that the number of  $(1/2, k)$ -quasiradial points for  $P$  is at least  $\Omega(n/k^2)$ . To see this, take all integer lattice points in a disk of radius  $\sqrt{n/\pi}$ , and note that the intersection of the line at infinity with every line  $y = (p/q)x$  with  $|p|, |q| < \sqrt{n}/(10k)$  is a  $(1/2, k)$ -quasiradial point.

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